PU.M.A. Vol. 8 (1997), No. 1, pp. 75–81

A note on bi- and quasi-ideals of semigroups, ordered semigroups*

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Abstract. Let S be a semigroup or an ordered semigroup. If S is regular, then the quasi-ideals and the bi-ideals of S coincide. The converse statement does not hold, in general. In regular ordered semigroups having a greatest element the quasi-ideal elements and the bi-ideal elements also coincide. If for an ordered semigroup S having a greatest element the quasi-ideal elements and the bi-ideal elements are the same, this does not mean that S is regular.

Mathematics Subject Classification. 06F05, 20M12

We have seen in the literature that a semigroup S is regular if and only if the quasi-ideals and the bi-ideals of S coincide. This is not true. The quasi-ideals are bi-ideals and, if S is regular, then the bi-ideals and the quasi-ideals of S coincide (cf. [8,9]). The converse statement does not hold, in general. The same for ordered semigroups: If an ordered semigroup S is regular, then the bi-ideals and the quasi-ideals of S are the same (cf. [4; Remark 2]). In particular, if S has a greatest element, that is, if it is a *poe* (in particular, *le*, $\vee e$)-semigroup, then the bi-ideal elements and the quasi-ideal elements of S also coincide. The converse statements do not hold, in general. By a *poe*-semigroup we mean an ordered semigroup (: *po*-semigroup) with a greatest

*Received: January 22, 1996

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element. An le ($\vee e$)-semigroup is an l ($\vee$)-semigroup with a greatest element [3]. A $\vee$-semigroup is a semigroup at the same time a semilattice under "$\vee$" such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all a, b, c. If the $\vee$-semigroup is a lattice, then it is called an l-semigroup [1; p. 323, 2; p. 153]. In this paper we give examples of non-regular poe, le, $\vee e$-semigroups for which the quasi-ideals (resp. the quasi-ideal elements) and the bi-ideals (resp. bi-ideal elements) are the same. We also give examples of non-regular po-semigroups and non-regular semigroups – without order – in which the quasi-ideals and the bi-ideals coincide. We are interested in finding examples with pure po, poe, $\vee e$-semigroups, that is po-semigroups which are not poe, poe-semigroups which are not $\vee e$ or le, $\vee e$-semigroups which are not le; also examples on poe (le)-semigroups. For the necessary definitions concerning the ordered semigroups we refer to [3,4] (cf. also [6]). For an easy way to check that the sets of the examples of this paper are ordered semigroups, we refer to [5, 7].

Example 1. The semigroup $S = \{a, b, c, d, e\}$ defined by the multiplication "." below is not regular ($e \neq exe\ \forall\ x \in S$).

$\cdot$	a	b	c	d	e
a	a	b	c	a	e
b	b	c	b	b	b
c	c	b	c	c	c
d	d	b	c	d	e
e	c	b	c	c	c

The quasi-ideals of S are the sets:

$\{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, b, c, d\}, \{b, c, e\}, \{a, b, c, e\}$, $\{b, c, d, e\}$ and S.

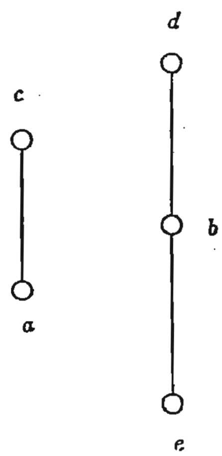
The bi-ideals of S coincide with the quasi-ideals.

Example 2. The po-semigroup $S = \{a, b, c, d, e\}$ defined by the multiplication "." and the order "$\leq$" below is not regular ($\nexists\ x \in S : d \leq dxd$) and the sets of bi-ideals and quasi-ideals of S coincide.

$\cdot$	a	b	c	d	e
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
e	e	e	e	e	e

$\leq = \{(a,a),(a,c),(b,b),(b,d),(c,c),(d,d),(e,b),(e,d),(e,e)\}$.
We give the covering relation " $\prec$ " and the figure of S.

$$\prec = \{(a,c),(b,d),(e,b)\}$$



The quasi-ideals of S are the sets:
$\{e\},\{b,e\},\{a,b,e\},\{a,b,c,e\},\{b,d,e\},\{a,b,d,e\}$ and S.
The bi-ideals of S are the same.

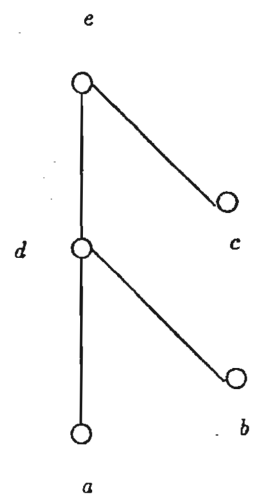
Example 3. The *poe*-semigroup $S = \{a,b,c,d,e\}$ defined by the multiplication "." and the order "$\leq$ " below is not regular ($(c,cec) \notin \leq$). The sets of bi-ideals and quasi-ideals of S coincide. The sets of bi-ideal elements and quasi-ideal elements of S coincide. This is not a $\vee e$-semigroup since, for example, $a(a \vee b) \neq a^2 \vee ab$.

$\cdot$	a	b	c	d	e
a	a	a	a	d	e
b	a	b	a	d	a
c	a	a	a	d	e
d	d	d	d	d	e
e	d	d	d	d	e

$$\leq \; = \; \{(a,a),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,e),\\ (d,d),(d,e),(e,e)\}.$$

We give the covering relation " $\prec$ " and the figure of S.

$$\prec = \{(a,d),(b,d),(c,e),(d,e)\}$$



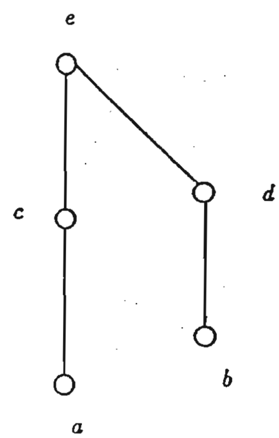
The quasi-ideals of S are the sets: $\{a,b,d\}$, $\{a,b,c,d\}$ and S.

The bi-ideals of S are the same.

The bi-ideal elements of S are the elements d and e. The elements d and e are also quasi-ideal elements of S. [So the sets of bi- and quasi-ideal elements of S coincide].

Example 4. The $\vee e$-semigroup $S = \{a,b,c,d,e\}$ defined by the multiplication "." and the figure below is not regular ($(c, cec) \notin\leq$). The sets of bi-ideals and quasi-ideals of S coincide. The sets of bi-ideal elements and quasi-ideal elements of S coincide.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	d	b	d	d	d
c	a	a	a	a	a
d	d	d	d	d	d
e	e	e	e	e	e

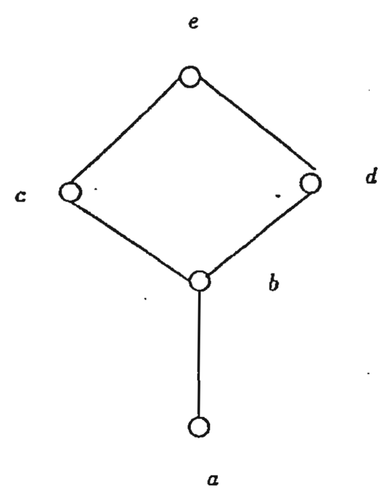


The bi-ideals of S, which coincide with the quasi-ideals of S, are the sets: $\{a\}, \{a,c\}, \{b,d\}, \{a,b,d\}, \{a,b,c,d\}$ and S.

The elements a, c, d, e are the bi-ideal elements of S, and each bi-ideal element is a quasi-ideal element, as well.

Example 5. The *le*-semigroup $S = \{a,b,c,d,e\}$ defined by the multiplication "." and the figure below is not regular ($(b, beb) \notin \leq$). The sets of bi-ideals and quasi-ideals of S coincide. The sets of bi-ideal elements and quasi-ideal elements of S coincide.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	a	c
d	d	d	d	d	d
e	d	d	e	d	e



The bi-ideals of S, which coincide with the quasi-ideals of S, are the sets: $\{a\}, \{a,c\}, \{b,d\}, \{a,b,d\}, \{a,b,c,d\}$ and S.

The elements a, b, c, d, e are the bi-ideal elements of S, and each bi-ideal element is a quasi-ideal element, as well.

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